October Math Gems

Problem of the week 25

§1 Problems

Problem 1.1. a, b, c, d satisfy the following system of equations

$$ab + c + d = 13$$

$$bc + d + a = 27$$

$$cd + a + b = 30$$

$$da + b + c = 17$$

Compute the value of $a + b + c + d$.

Problem 1.2. Suppose that we have the following set of equations

$$\log_2 x + \log_3 x + \log_4 x = 20$$

$$\log_4 y + \log_9 y + \log_{16} y = 16$$

Compute $\log_x y$.

Problem 1.3. If the function f satisfy the following relation

$$f\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3}$$

Then compute $f(4)$

Problem 1.4. In the following functional equation, solve for $f(x)$

$$f\left(x + \sqrt{x^2 + 1}\right) = \frac{x}{x + 1}$$

Problem 1.5. Let a, b, and c be distinct nonzero real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$$

Prove that $|abc| = 1$

Problem 1.6. Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$

Problem 1.7. Find all real numbers x satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1$$

Problem 1.8. If $z = x - iy$ and $z^{\frac{1}{3}} = p + iq$, then compute

$$\frac{x/p + y/q}{p^2 + q^2}$$

.

Problem 1.9. If

$$\frac{\log(a)}{b-c} = \frac{\log(b)}{c-a} = \frac{\log(c)}{a-b}$$

Then compute $a^a b^b c^c$.

Problem 1.10. If the coefficients of x^{-2} and x^{-4} in the expansion of

$$\left(x^{\frac{1}{3}} + \frac{1}{2x^{\frac{1}{3}}}\right) \quad (x > 0)$$

are m and n respectively, then compute $\frac{m}{n}$.

Problem 1.11. The coefficients of x^{50} in the expansion of

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}$$

is [1002]50. Prove or Disprove.

Problem 1.12. Suppose x and y are nonzero real numbers simultaneously satisfying the following system of equations

$$x + \frac{2018}{y} = 1000$$

$$\frac{9}{x} + y = 1$$

Find the maximum possible value of $x + 1000y$.

Problem 1.13. From the following system of equations

$$x^2 - y^2 = 9$$

$$xy = 3$$

The value of of $x + y$ can be written in the form of $\pm\sqrt{\sqrt{a} + b}$, then find the values of a and b.

Problem 1.14. Determine the domain of the function

$$g(x) = \cot^{-1}\left(\frac{x}{\sqrt{x^2 - \lfloor x^2 \rfloor}}\right)$$

Problem 1.15. If a and b are two real numbers, then show that

$$\left|\frac{a+bi}{b+ai}\right| = 1$$

Problem 1.16. Let a, b, c be distinct real numbers. Prove the the following equality cannot hold:

$$\sqrt[3]{a-b} + \sqrt[b]{b-c} + \sqrt[3]{c-a} = 0$$

Problem 1.17. let r be a real number such that

$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3$$

Determine the value of

$$r^3 + \frac{1}{r^3}$$

Problem 1.18. let $x, y, z > 0$. Prove that

$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \geq \frac{9}{x+y+z}$$

Problem 1.19. Let m, n be positive integers with $m < n$. Find the a closed form for the sum

$$\frac{1}{\sqrt{m}+\sqrt{m+1}} + \frac{1}{\sqrt{m+1}+\sqrt{m+2}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}}$$

Problem 1.20. Let a and b be distinct real numbers. Solve the following equation

$$\sqrt{x-b^2} - \sqrt{x-a^2} = a - b$$
